

Instantons in Burgers Equation

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Abstract

The instanton solution for the forced Burgers equation is found. This solution describes the exponential tail of the probability distribution function of velocity differences in the region where shock waves are absent. The results agree with the one found recently by Polyakov, who used the operator product conjecture. If this conjecture is true, then our WKB asymptotics of the Wyld functional integral is exact to all orders of perturbation expansion around the instanton solution. We explicitly checked this in the first order. We also generalized our solution for the arbitrary dimension of Burgers (=KPZ) equation. As a result we found the angular dependence of the velocity difference PDF.

There are two complementary views of the turbulence problem. One could regard it as kinetics, in which case the time dependence of velocity probability distribution function (PDF) must be studied. The Wyld functional integral describes the correlations functions of the velocity field in this picture.

Another view is the Hopf (or Fokker-Planck) approach, where the equal time PDF is studied. For the random force distributed as white noise in time, the closed functional equations (Fokker-Planck equation) can be derived. In the case of thermal noise the Boltzmann distribution can be derived as an asymptotic solution of this equation.

One of the authors [1] reduced the Hopf equations for the full Navier-Stokes equation to the one dimensional functional equation (loop equation). The WKB solutions of this equation were studied, leading to the area law for the velocity circulation PDF.

In the recent paper by Polyakov [2] a similar method of solving the randomly driven Burgers equation was proposed. It reduced the problem of computations of their correlation functions to the solution of a certain partial differential equation. This equation for the velocity difference PDF can be explicitly solved.

The derivation of the Polyakov equation was based on the conjecture of the existence of the operator product expansion.

On the other hand, it was recently understood in the paper [3] by Falkovich, Kolokolov, Lebedev and one of us that the Wyld functional integral provides the general method of computation of the PDF tails. One has to find an *instanton* – the minimum of the action in the Wyld functional integral.

This instanton is not the same as the solution of the classical equation in the usual sense. The force is present, and it acts in a self-consistent way, as required by minimization of the action. This force is no longer random, it is adjusted to provide the large fluctuation of velocity field under consideration.

It was shown in [3] that the PDF for the passive scalar advection in the Gaussian velocity field is asymptotically described by an instanton, with the spatially homogeneous strain.

Here we find instanton in the Burgers equation, in the presence of finite viscosity. The result which we obtain in the turbulent limit (vanishing viscosity) coincides with that of Polyakov which gives an indirect confirmation to his OPE conjecture.

We start with the randomly driven Burgers equation.

$$u_t + uu_x - \nu u_{xx} = f(x, t) \quad (1)$$

where the force $f(x, t)$ is a gaussian random field with a pair correlation function

$$\langle f(x, t)f(y, t') \rangle = \delta(t - t')\kappa(x - y) \quad (2)$$

The Wyld functional integral has the following form (see, for example, [4]).

$$\begin{aligned} \int \mathcal{D}f \exp \left(-\frac{1}{2} \int dx dy dt f(x, t) D(x - y) f(y, t) \right) = \\ \int \mathcal{D}f \mathcal{D}u \delta[u_t + uu_x - \nu u_{xx} - f] \exp \left(-\frac{1}{2} \int dx dy dt f(x, t) D(x - y) f(y, t) \right) = \\ \int \mathcal{D}f \mathcal{D}u \mathcal{D}\mu \exp \left(i \int dx dt \mu (u_t + uu_x - \nu u_{xx} - f) - \frac{1}{2} \int dx dy dt f(x, t) D(x - y) f(y, t) \right) = \\ \int \mathcal{D}u \mathcal{D}\mu \exp \left(i \int dx dt \mu (u_t + uu_x - \nu u_{xx}) - \frac{1}{2} \int dx dy dt \mu(x, t) \kappa(x - y) \mu(y, t) \right) \end{aligned} \quad (3)$$

Here we started with the obvious functional integral for the force, where D is the function inverse to κ . To change the variable of integration from f to u we inserted into the functional integral the identity

$$\mathcal{N} = \int \mathcal{D}u \delta[u_t + uu_x - \nu u_{xx} - f] \quad (4)$$

where \mathcal{N} is just a number. Dropping that number as an unimportant constant, we removed the delta function at the expense of introducing a “conjugated” variable μ and evaluated the integral over f to arrive at the final expression in (3).

It is not so obvious that the integral in (4) is equal to a pure number because of the determinant $\det \left(\frac{\delta f}{\delta u} \right)$ arising in its computation. It is possible, however, to prove (4) using the causality argument (see [4]).

Thus the initial problem of computing the correlation functions of Burgers equation is reduced to the field theory with the action

$$S = -i \int \mu(u_t + uu_x - \nu u_{xx}) + \frac{1}{2} \int dx dy dt \mu(x, t) \kappa(x - y) \mu(y, t) \quad (5)$$

Here we are going to study the correlation function

$$\langle \exp \{ \lambda_0 (u(\rho_0/2) - u(-\rho_0/2)) \} \rangle = \int \mathcal{D}u \mathcal{D}\mu \exp \{ \lambda_0 (u(\rho_0/2) - u(-\rho_0/2)) - S \} \quad (6)$$

whose Laplace Transform gives us the two point probability distribution (see [2]). We will often refer to the expression we have in the exponential as the action

$$S_{\lambda_0} = S - \lambda_0 (u(\rho_0/2) - u(-\rho_0/2)) \quad (7)$$

There are no general methods to compute the functional integrals like (6) exactly. The most straightforward approach would be to expand the exponential in functional integral in powers of the nonlinear term $\mu u u_x$. By doing so we will just reproduce the well known Wyld's diagram technique (see [5]). The attempts to use this technique to describe turbulence always failed because we are interested in the limit $\nu \rightarrow 0$ when nonlinear term dominates the functional integral. The absence of a large parameter makes the task of computing this functional integral by using perturbation theory hopeless.

Nevertheless, if we are interested in computing the large λ_0 behavior of the correlation function in (6), we can use λ_0 itself as a large parameter. Then the integral will be dominated by its saddle point, or by the solutions of the equations of motion for the action (7). All we have to do is to find those solutions and compute the value of the action S_{inst} on those solutions. The answer will be given by

$$\langle \exp \{ \lambda_0 (u(\rho_0/2) - u(-\rho_0/2)) \} \rangle = \exp \{ -S_{\text{inst}}(\lambda_0) + S_{\text{inst}}(0) \}. \quad (8)$$

If we want, we can then further expand the integral in powers of $1/\lambda_0$ using the perturbation theory around those solutions. We will call this method WKB approximation and the solutions instantons using the names borrowed from quantum field theory.

To that effect, let us write down the equations of motion corresponding to the action (7). They are

$$u_t + u u_x - \nu u_{xx} = -\imath \int dy \kappa(x-y) \mu(y) \quad (9)$$

$$\mu_t + u \mu_x + \nu \mu_{xx} = -\imath \lambda_0 \left\{ \delta \left(x - \frac{\rho_0}{2} \right) - \delta \left(x + \frac{\rho_0}{2} \right) \right\} \delta(t) \quad (10)$$

To solve these equations, let us first notice that the only role the right hand side of (10) plays is giving the field μ a finite discontinuity at $t = 0$. It is also easy to see that $\mu(t) = 0$ for $t > 0$. This is because μ feels a negative viscosity so any solution which is nonzero at $t > 0$ will become singular¹. Thus the field μ can be evaluated at $t = -0$ to be

$$\mu(t = -0) = \imath \lambda_0 \left\{ \delta \left(x - \frac{\rho_0}{2} \right) - \delta \left(x + \frac{\rho_0}{2} \right) \right\} \quad (11)$$

while it is zero at all later moments of time. It is therefore convenient to speak of the field μ propagating backwards in time starting from its initial value given by (11).

¹ Alternatively, one can argue that the integrals in (3) are defined only for $t < 0$. Those arguments use a striking similarity between (3) and a Feynman path integral for a quantum mechanical system with the coordinates u and momenta μ to define (6) as a wave function in the momentum representation. Then the conditions (11) become obvious.

If we try to propagate (11) back in time, we discover that we have to deal with two phenomena governed by the second and third terms of (10). One of them is just a motion of the initial conditions as dictated by the velocity u . The other is the “smearing” of the initial delta function distributions in (11) due to the viscosity.

However, it can be shown by a direct computation that the smearing does not change the value of the action on the instanton as long as the viscosity is not very large. We will construct a solution which takes into account the viscosity in the end of the paper. For now we will just drop the viscosity term to arrive at a simplified equation

$$\mu_t + u\mu_{xx} = 0 \quad (12)$$

Since all this equation can do is moving the δ function-like singularities around (and changing their heights by compressing them), it is clear that the solution of (12) with the boundary conditions given by (11) is just

$$\mu(t) = \lambda(t) \left\{ \delta \left(x - \frac{\rho(t)}{2} \right) - \delta \left(x + \frac{\rho(t)}{2} \right) \right\} \quad (13)$$

with the boundary conditions

$$\lambda(0) = \lambda_0, \quad \rho(0) = \rho_0 \quad (14)$$

Now let us leave the equation (10) for a while and study (9). A natural thing to do is to substitute (13) into the right hand side of (9).

We obtain

$$u_t + uu_x - \nu u_{xx} = \lambda \left\{ \kappa \left(x - \frac{\rho}{2} \right) - \kappa \left(x + \frac{\rho}{2} \right) \right\} \quad (15)$$

To proceed further we need to know κ . Let us assume, following [2], that $\kappa(x)$ is a slow varying even function of x which behaves as

$$\kappa(x) \approx \kappa(0) - \frac{\kappa_0}{2}x^2, \quad |x| \ll \sqrt{\frac{\kappa(0)}{\kappa_0}} \equiv L \quad (16)$$

and quickly turns into zero when $|x| \gg L$. The interval L characterizes the range of the random force and we will work only there, that is we suppose that ρ also lies within this interval. It is clear then that the contribution to the action (7) comes only from the interval L (compare with (13)). So we do not have to know the velocity beyond that interval. There we use (16) to obtain

$$u_t + uu_x - \nu u_{xx} = \lambda \rho \kappa_0 x \quad (17)$$

Notice that $\kappa(0)$ dropped out.

(17) is a Burgers equation with a linear force. It is easy to solve such an equation. We have to look for the solution in terms of a linear function

$$u(x, t) = \sigma(t)x \quad (18)$$

which leads to

$$\frac{d\sigma}{dt} + \sigma^2 = \kappa_0 \lambda \rho \quad (19)$$

Notice that the viscosity term did not contribute. That does not mean that the viscosity is not important at all. For $x \gg \rho$ the force in the (15) becomes zero and the viscosity there is important to make the velocity go to zero at the infinity. However, in the region $x \propto \rho$ which is the one we study the viscosity term can be dropped.

Now we can use (18) and (13) to solve (12). A direct substitution leads to

$$\begin{aligned}\frac{d\lambda}{dt} &= \lambda\sigma \\ \frac{d\rho}{dt} &= \rho\sigma\end{aligned}\tag{20}$$

These can be solved in terms of the function

$$R(t) = \exp\left(\int_0^t dt' \sigma(t')\right)$$

to give

$$\begin{aligned}\lambda &= \lambda_0 R \\ \rho &= \rho_0 R\end{aligned}\tag{21}$$

while R itself satisfies, by virtue of (19), the equation

$$\frac{d^2 R}{dt^2} = \kappa_0 \rho_0 \lambda_0 R^3\tag{22}$$

The last equation has to be solved with the boundary condition $R(-\infty) = 0$ otherwise the action (7) will not be finite. The solution is given by

$$R = \frac{1}{1 - \sqrt{\frac{\kappa_0 \rho_0 \lambda_0}{2}} t}\tag{23}$$

So we have found the instanton solution for the equations (9), (10). Notice that it is the *only* solution of the equations of motion, so we do not have to sum over different instantons.

Now it is a matter of a simple computation to find the action on the instanton. We collect everything together and substitute (23), (21), (13), and (18) back to (7) to get

$$S_{\text{inst}} = -\frac{\sqrt{2\kappa_0}}{3}(\lambda_0 \rho_0)^{\frac{3}{2}}\tag{24}$$

while the correlation function we were studying is

$$\langle \exp \{ \lambda_0 (u(\rho_0/2) - u(-\rho_0/2)) \} \rangle = \exp \left(\frac{\sqrt{2\kappa_0}}{3} (\lambda_0 \rho_0)^{\frac{3}{2}} \right)\tag{25}$$

This is the same answer as the one obtained in [2]. We want to emphasize, however, that we obtained it without any conjectures and only as an asymptotics for $|\lambda_0| \gg 1$.

We are not going to discuss the physical implications of (25) referring instead to the papers [2] and [6].

Now we return to the question of why we can drop the viscosity in (10). Namely, we just construct the solution of (10) with the viscosity. To do that, it is convenient to Fourier transform it (taking into account that the velocity u is a linear function of x).

$$\frac{\partial \mu(p)}{\partial t} - \sigma \frac{\partial}{\partial p} (p \mu(p)) - \nu p^2 \mu = 0 \quad (26)$$

Then the solution of (26) can be found as a direct generalization of (13).

$$\mu(p) \propto \lambda(t) \sin \left(\frac{p \rho(t)}{2} \right) \exp \left(-\beta(t) p^2 \right) \quad (27)$$

Here we had to introduce the new variable $\beta(t)$ which measures the speed of smearing of the solution with the evident initial condition $\beta(0) = 0$. Substituting (27) into (26) we reproduce the equations (21) for ρ and λ with the additional equation for β

$$\beta_t - 2\sigma\beta + \nu = 0 \quad (28)$$

with the solution

$$\beta(t) = \frac{\nu}{3\omega} \frac{(1 - \omega t)^3 - 1}{(1 - \omega t)^2} \quad (29)$$

where $\omega = \sqrt{(\kappa_0 \rho_0 \lambda_0)/2}$.

The smearing, however, has no influence whatsoever on the velocity. To see that, we substitute (27) into (9) and arrive back at (17). In other words, the variable σ still satisfies the same equation (19). A simple argument given below shows that the value of the instanton action depends only on the final value of the velocity of the instanton solution which, as we just showed, does not depend on the viscosity.

We must remember, though, that we should not allow μ to spread beyond L interval, or more precisely $L > \sqrt{\beta}$. β can become arbitrarily large for large negative times, but the characteristic time interval which contributed to the computation of the instanton action is

$$t_{\text{inst}} = \frac{1}{\omega} \quad (30)$$

So all we have to do is to make sure that $\beta(t_{\text{inst}}) < L^2$ or

$$L^2 > \frac{\nu}{\omega} \quad (31)$$

This is the condition which viscosity must satisfy for (25) to be correct and independent of viscosity.

So far we cannot claim that (25) is an exact answer. It is just a leading asymptotic if $|\lambda_0|$ is a large number. It might be important to estimate the next order contribution to the (25) especially in view of the claim made in [2] that (25) is actually exact.

To do that it is convenient to introduce the quantity

$$\frac{\partial}{\partial \lambda_0} \log \langle \exp \{ \lambda_0 (u(\rho_0/2) - u(-\rho_0/2)) \} \rangle = \frac{\int \mathcal{D}u \mathcal{D}\mu (u(\rho_0/2) - u(-\rho_0/2)) \exp(-S_{\lambda_0})}{\int \mathcal{D}u \mathcal{D}\mu \exp(-S_{\lambda_0})} \quad (32)$$

It is easy to expand this quantity around the instanton solution. Writing $u = u_{\text{inst}} + \tilde{u}$, $\mu = \mu_{\text{inst}} + \tilde{\mu}$ we arrive for (32) at

$$u_{\text{inst}} \left(\frac{\rho_0}{2} \right) - u_{\text{inst}} \left(-\frac{\rho_0}{2} \right) + \frac{\int \mathcal{D}\tilde{u} \mathcal{D}\tilde{\mu} (\tilde{u}(\rho_0/2) - \tilde{u}(-\rho_0/2)) \exp \{-S_{\lambda_0}(u_{\text{inst}} + \tilde{u}, \mu_{\text{inst}} + \tilde{\mu})\}}{\int \mathcal{D}\tilde{u} \mathcal{D}\tilde{\mu} \exp \{-S_{\lambda_0}(u_{\text{inst}} + \tilde{u}, \mu_{\text{inst}} + \tilde{\mu})\}} \quad (33)$$

The first term of the expression corresponds to the instanton contribution to the action. We see that it depends only on the value of the instanton solution at $t = 0$, justifying our argument that the viscosity does not contribute to the answer. We can easily compute this term by substituting the known instanton solution to get

$$\sqrt{\frac{\kappa_0 \lambda_0}{2}} \rho_0^{\frac{3}{2}} \quad (34)$$

which is of course compatible with (25).

Let us proceed to estimate the second term in (33). It is not difficult to see that if we expand $S_{\lambda_0}(u_{\text{inst}} + \tilde{u}, \mu_{\text{inst}} + \tilde{\mu})$ in powers of \tilde{u} and $\tilde{\mu}$ up to a second order, the contribution from that will be zero as S will be an even function of \tilde{u} and $\tilde{\mu}$. However, due to the presence of the third order terms in the action, it is not clear if the higher order terms are also zero. If they are nonzero, then a dimensional argument shows that the next term in (34) is of the order $(1/\lambda_0)^2$. Really, then the expanded action looks like

$$S(u_{\text{inst}} + \tilde{u}, \mu_{\text{inst}} + \tilde{\mu}) = S_{\text{inst}} + S_{\text{quad}} + S_{\text{int}} = S_{\text{inst}} - \imath \int \tilde{\mu}(\tilde{u}_t - \nu \tilde{u}_{xx}) + \quad (35)$$

$$\frac{1}{2} \int dx dy dt \tilde{\mu}(x, t) \kappa(x - y) \tilde{\mu}(y, t) - \imath \int \tilde{\mu} \frac{\partial}{\partial x} (\tilde{u} u_{\text{inst}}) - \imath \int \mu_{\text{inst}} \tilde{u} \tilde{u}_x + S_{\text{int}}$$

S_{quad} meaning the quadratic part of the expansion and S_{int} being the interaction term, $S_{\text{int}} = -\imath \int \tilde{\mu} \tilde{u} \tilde{u}_x$. We need to take into account the interaction in the first order of perturbation theory. That is we need to evaluate

$$\langle \tilde{u}(\rho_0) S_{\text{int}} \rangle \quad (36)$$

understanding the average in the sense of the action S_{quad} . While the “honest” computation would require the computation of the Green’s functions for that action, we can estimate the value of (36) by noting that $u_{\text{inst}} \propto \sqrt{\lambda_0}$, $\mu_{\text{inst}} \propto \lambda_0$ and

$$S_{\text{quad}} \propto \lambda_0 \tilde{u}^2 + \sqrt{\lambda_0} \tilde{\mu} \tilde{u} \quad (37)$$

Therefore the typical fluctuations are $\delta \tilde{u} \propto \frac{1}{\sqrt{\lambda_0}}$ and $\delta \tilde{\mu} \propto 1$. The expression (36) involves three u and one μ , so we could expect it to be of the order $1/\lambda_0^{\frac{3}{2}}$. However, there is also an integration over t involved. That integration gives us $1/\omega \propto 1/\sqrt{\lambda_0}$. Altogether we arrive at $1/\lambda_0^2$.

In other words, it could be expected that the correction to the answer has the form

$$\langle \exp \{ \lambda_0 (u(\rho_0/2) - u(-\rho_0/2)) \} \rangle \propto \exp \left(\frac{\sqrt{2\kappa_0}}{3} (\lambda_0 \rho_0)^{\frac{3}{2}} + \frac{\text{const}}{\lambda_0} \right) \quad (38)$$

However, this question requires much deeper investigations which we leave for future work. The formula (38) is just a dimensional estimation. The Green's functions for S_{quad} have to be constructed for the corrections to the answer (25) to be computed reliably. That has not been done yet.

The analysis of this paper can easily be extended for the case of more than one dimension. The analog of the equations (9) and (10) will be

$$\frac{\partial u_i}{\partial t} + \left(u_j \frac{\partial}{\partial x_j} \right) u_i - \nu \Delta u = -\imath \int dy \kappa_{ij}(x-y) \mu_j(y) \quad (39)$$

$$\frac{\partial \mu_i}{\partial t} + \frac{\partial}{\partial x_j} (u_j \mu_i) - \mu_j \frac{\partial u_j}{\partial x_i} + \nu \Delta \mu = -\imath \lambda_{0i} \left\{ \delta \left(x - \frac{\rho_0}{2} \right) - \delta \left(x + \frac{\rho_0}{2} \right) \right\} \delta(t) \quad (40)$$

The solution of these equations is a direct generalization of (13), or

$$\mu_i(t) = \imath \lambda_i(t) \left\{ \delta \left(x - \frac{\rho(t)}{2} \right) - \delta \left(x + \frac{\rho(t)}{2} \right) \right\} \quad (41)$$

with

$$\lambda_i(0) = \lambda_{0i}, \quad \rho_i(0) = \rho_{0i} \quad (42)$$

The further progress depends on the tensorial structure of κ_{ij} which is just a correlation function

$$\langle f_i(x, t) f_j(y, t') \rangle = \kappa_{ij}(x-y) \delta(t-t') \quad (43)$$

A natural thing to assume would be that the force is a gradient of something, that is $f_i = \partial_i \Phi$, in which case

$$\kappa_{ij}(x) \approx \kappa_{ij}(0) - \frac{\kappa_0}{2} (x^2 \delta_{ij} + 2x_i x_j) \quad (44)$$

A direct generalization of the velocity ansatz is

$$u_i = \sigma_{ij} x_j \quad (45)$$

and the equations (19) and (20) turn into

$$\begin{aligned} \frac{d\lambda_i}{dt} &= \lambda_j \sigma_{ji} \\ \frac{d\rho_i}{dt} &= \sigma_{ij} \rho_j \\ \frac{d\sigma_{ij}}{dt} + \sigma_{ik} \sigma_{kj} &= \kappa_0 (\lambda_i \rho_j + \rho_i \lambda_j + \delta_{ij} \lambda_k \rho_k) \end{aligned} \quad (46)$$

σ can actually be eliminated from those equations to give us an analog of (22),

$$\begin{aligned} \frac{d^2 \lambda_i}{dt^2} &= \kappa_0 (\rho_i \lambda^2 + 2\lambda_i \lambda_k \rho_k) \\ \frac{d^2 \rho_i}{dt^2} &= \kappa_0 (\lambda_i \rho^2 + 2\rho_i \lambda_k \rho_k) \end{aligned} \quad (47)$$

While a general solution of those equations is rather difficult to find, it is possible to find the action on the solution by analyzing the corresponding Hamilton-Jacobi equation. To do that, we note that the equations (47) are hamiltonian with the hamilton function

$$H = \frac{d\lambda_i}{dt} \frac{d\rho_i}{dt} - \frac{\kappa_0}{2} \left(\rho^2 \lambda^2 + 2(\lambda_k \rho_k)^2 \right) \quad (48)$$

The (time independent) instanton action S clearly satisfies the equation ²

$$\frac{\partial S}{\partial \lambda_i} \frac{\partial S}{\partial \rho_i} - \frac{\kappa_0}{2} \left(\rho^2 \lambda^2 + 2(\lambda_k \rho_k)^2 \right) = 0 \quad (49)$$

By rescaling the time and the variables ρ and λ in (47) we can show that the action S has the following initial condition dependence

$$S = \sqrt{\frac{\kappa_0}{2}} (\rho_0 \lambda_0)^{\frac{3}{2}} f(\cos \varphi) \quad (50)$$

where φ is the angle between the vectors of initial conditions λ_{0i} and ρ_{0i} . This ansatz coincides with the one dimensional answer (24) up to a nontrivial function of the angle $f(\cos \varphi)$ which we would like to determine. Plugging the ansatz into the Hamilton-Jacobi equation we obtain the equation for $f(\cos \varphi)$

$$\frac{9}{4} z f^2 + 3 f f'(1 - z^2) + f'^2(-z + z^3) = 1 + 2z^2 \quad (51)$$

where $z = \cos(\varphi)$. This first order differential equation has to be solved with the boundary condition

$$f(1) = \frac{2}{\sqrt{3}} \quad (52)$$

which follows directly from the equation (51) but also can be computed by solving the equation of motion for $\varphi = 0$. We can find the function f as a series in powers of $1 - z$. It turns out there are two solutions

$$\begin{aligned} f(z) &= \frac{2}{\sqrt{3}} - \frac{\sqrt{3} + \sqrt{11}}{4} (1 - z) + \frac{5\sqrt{33} - 61}{32(3\sqrt{3} - 2\sqrt{11})} (1 - z)^2 + \dots \\ f(z) &= \frac{2}{\sqrt{3}} + \frac{\sqrt{11} - \sqrt{3}}{4} (1 - z) - \frac{5\sqrt{33} + 61}{32(3\sqrt{3} + 2\sqrt{11})} (1 - z)^2 + \dots \end{aligned} \quad (53)$$

The equation (51) does not tell us which of these two to choose. We have to match (53) with the solution of (47). Those equations cannot be solved in general, but there is a way to find their solution if the angle φ is close to zero which should be enough to determine $f'(1)$ and therefore to choose the right action.

²We would like to note that the equation (49) follows from the master equation of [2] if the viscosity is completely neglected. The authors are grateful to A. Polyakov for pointing that out.

To do that, we note that the motion represented by (47) is essentially two dimensional, with all the motion confined to the λ_0, ρ_0 plane. Then we represent λ as a two-vector (λ_1, λ_2) , while $\rho = (\lambda_1, -\lambda_2)$. The equations (47) turn into (we choose the units where $\kappa_0 = 1$)

$$\begin{aligned}\frac{d^2\lambda_1}{dt^2} &= 3\lambda_1^3 - \lambda_1\lambda_2^2 \\ \frac{d^2\lambda_2}{dt^2} &= -3\lambda_2^3 + \lambda_2\lambda_1^2\end{aligned}\tag{54}$$

We choose the boundary conditions $\lambda_0 = 1, \rho_0 = 1$. If $\varphi = 0$ then $\lambda_2 = 0$ while λ_1 satisfies the equation

$$\frac{d^2\lambda_1}{dt^2} = 3\lambda_1^3\tag{55}$$

hence

$$\lambda_1 = \frac{1}{1 - \omega t}, \quad \omega = \sqrt{\frac{3}{2}}\tag{56}$$

Now if φ is a small number then $\lambda_2 \ll \lambda_1$ and it satisfies the approximate equation

$$\frac{d^2\lambda_2}{dt^2} = \frac{\lambda_2}{(1 - \omega t)^2}\tag{57}$$

with the solution

$$\lambda_2 = \frac{C}{(1 - \omega t)^\alpha}, \quad \alpha = \frac{-3 \pm \sqrt{33}}{6}\tag{58}$$

In particular, for $t = 0$,

$$\frac{d \log \lambda_2}{dt} = \frac{-\sqrt{3} \pm \sqrt{11}}{2\sqrt{2}}\tag{59}$$

That last quantity can also be evaluated if we know the action S . For $t = 0$ we obtain

$$\frac{d\rho_i}{dt} = \frac{\partial S}{\partial \lambda_i} = \frac{3\lambda_i}{2\sqrt{2}}f(z) + \frac{f'(z)}{\sqrt{2}}(\rho_i - z\lambda_i)\tag{60}$$

which translates to the language of λ_2 ($\lambda_2 \ll \lambda_1$ and $z \approx 1$) as

$$\frac{d \log \lambda_2}{dt} = \sqrt{2}f'(1) - \sqrt{\frac{3}{2}}\tag{61}$$

Comparing with (59) we get

$$f'(1) = \frac{\sqrt{3} \pm \sqrt{11}}{4}\tag{62}$$

Now we need to choose the plus sign in all the above formulas as we want α to be positive. Otherwise our action will correspond to the solution growing at $t \rightarrow -\infty$. That makes us choose the first $f(z)$ in (53).

We would also like to note that according to the equation (51) $f(-1) = \imath f(1)$ which can be checked directly by solving the equations of motion at $\varphi = \pi$. Moreover, it can be seen from (51) that $\imath f(-z)$ is its solution if $f(z)$ is a solution. So we believe there should be

some kind of a crossover where the real solution becomes purely imaginary. The fact that the correlation function of a real quantity becomes imaginary should not disturb us. It means that the correlation function we are computing may not exist for a certain value of λ_i and can only be understood in the sense of analytic continuation. Apparently, the logarithm of probability distribution function which is obtained by a Legendre transform of the action we found must remain real. One could perform this transform term by term in our expansion.

Summarizing everything, the answer for the D-dimensional case is given by

$$\left\langle \exp \left\{ \lambda_i \left[u_i \left(\frac{\rho}{2} \right) - u_i \left(-\frac{\rho}{2} \right) \right] \right\} \right\rangle = \exp \left\{ \sqrt{\frac{\kappa_0}{2}} (\rho \lambda)^{\frac{3}{2}} f(\cos \varphi) \right\} \quad (63)$$

while a Legendre transform of the action will give us the probability distribution function in the form

$$\left\langle \delta \left\{ u_i - \left[u_i \left(\frac{\rho}{2} \right) - u_i \left(-\frac{\rho}{2} \right) \right] \right\} \right\rangle \approx \exp \left\{ -\frac{2}{9\kappa_0} \frac{u^3}{\rho^3} \left(1 + \frac{9(\sqrt{33}-1)}{(9-\sqrt{33})^2} \psi^2 + \dots \right) \right\} \quad (64)$$

where ψ is the angle between u_i and ρ_i , $\psi \ll 1$, and $u/\rho \rightarrow +\infty$.

In conclusion, we would like to say that we have showed by a simple computation that WKB calculations are very useful to understand the behavior of randomly driven Burgers equation and we hope they will be found useful in other problems of turbulence as well. The instanton we found has a spacial homogeneous strain (the velocity was a linear function in the inertial interval) and we suspect it to be a general feature of the instantons in the turbulence problem.

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